Transmission Line Model with Frequency-Dependent and Nonuniform Parameters in Frequency and Time Domain

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Abstract— The paper presents a fast and effective method of modeling a nonuniform and dispersive interconnect by means of S-parameters. The paper presents an approach based on the method of successive approximations, but taking into account the dependence on the frequency of line parameters. The concept is to use a rational approximation of the per-unit-length parameter of the line calculated for each frequency. An example of the Bessel dispersive transmission line has been considered.

Keywords — Interconnect, VLSI, Scattering Parameters, Transmission Line, Spice

I. INTRODUCTION

Modeling of transmission lines in the time-domain is an ongoing challenge for the people involved in the simulation of integrated circuits and/or printed circuit boards at high frequency. The literature on this subject is very rich and can be found e.g. in [2,3,4]. Among many methods and approaches we would like to focus on two, which include further references. In the first paper [2], the author presents an approach based on dyadic Green's function and vector fitting of per-unit-length impedance and admittance of transmission line to obtain a Z matrix of transmission line as a two-port. The line impedance and admittance are the sums of rational functions of complex frequency s, which facilitates the transformation to the time-domain and modeling in SPICE. The biggest problem is the necessity to take into account a large number of terms in every entry of the mentioned Z matrix. In [3], the same author has extended the above approach to weakly nonuniform transmission lines. In that case the author used results obtained for uniform case and parametric macromodeling to obtain the approximate Z matrix of the line. In both papers, the presented approach has been extended to the case of a multiconductor line. On the other hand in paper [4], a method was developed to convert of differential telegrapher's equations into integral equations and next to solve them using the method of successive approximation. In that approach, we obtain a first order approximation of the solution in a simple analytical form which is valid for low loss transmission lines. The drawback of that approach was not including the skin effect and dielectric dispersion.

This paper presents an improved version of the approach based on the method of successive approximations [4], taking into account the line parameter dependence on the frequency and longitudinal coordinate. For this purpose, as in [2,3], we use the concept of rational approximation of per-unit-length parameters of the line in the frequency domain. Our approach is based on scattering parameters of the transmission line. Such parameters for both frequency and time domains was obtained in [1] for a multiconductor but only uniform line. In this paper, we extended this approach to the case of a nonuniform, frequency-dependent single transmission line.

The paper is organized as follows. The next section presents the integral equations approach to the dispersive transmission line. In the third section, we employ the method of successive approximation to calculate the scattering parameters of a nonuniform transmission line. In the fourth section we present, the calculations for the Bessel frequency dependent transmission line. We conclude in the last section.

II. TELGRAPHER'S EQUATIONS IN INTEGRAL FORM

A. *Telegrapher's equations for a dispersive nonuniform transmission line*

The equations for a nonuniform, dispersive transmission line are the following:

$$-\frac{dV(s,y)}{dz} = Z(s)r(z)\,I(s,z), \qquad -\frac{dI\,(s,y)}{dz} = Y(s)g(z)V(s,z), \tag{1}$$

where

$$Z(s) = Z_o(s) + Z_1(s), Y(s) = Y_o(s) + Y_1(s),$$
$$Z_o(s) = R + sL, \qquad Y_o(s) = G + pC$$
$$Z_1(s) = \sum_{m=1}^{N_z} \frac{R_m^z}{s + p_m^z}, Y_1(s) = \sum_{m=1}^{N_y} \frac{R_m^y}{s + p_m^y}$$
$$z_1 \le z \le z_2, \qquad d = z_2 - z_1$$

d-length of the line
r(z), g(z)- transmission line taper.

In (1) Z_1 and Y_1 have rational form of per-unit-length impedance and admittance of the transmission line obtained as in [2] by means of the vector fitting technique [5]. The next step is introducing current waves instead of voltage and current into the transmission line equations (1). It is done, similarly as in [4], by transformations:

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$$\begin{bmatrix} I_- \\ I_+ \end{bmatrix}(z,s) = \boldsymbol{S}(z)\begin{bmatrix} V \\ I \end{bmatrix}(z,s), \boldsymbol{S} = \frac{1}{2}\begin{bmatrix} \sqrt{Y_c} & -\sqrt{Z_c} \\ \sqrt{Y_c} & \sqrt{Z_c} \end{bmatrix}$$
$$Y_c = \sqrt{\frac{g(z)}{r(z)}} = f_c(z)^{-1}. \tag{2}$$

Using transformation (2) we can pass to (3):

$$-\frac{d}{dz}\begin{bmatrix} I_- \\ I_+ \end{bmatrix} = \left\{\boldsymbol{S}\frac{d\boldsymbol{S}^{-1}}{dz} + \boldsymbol{S}\begin{bmatrix} 0 & Z \\ Y & 0 \end{bmatrix}\boldsymbol{S}^{-1}\right\}\begin{bmatrix} I_- \\ I_+ \end{bmatrix}. \tag{3}$$

Equations (3), after differentiation and simple algebraic operations, take the following scalar form:

$$-\frac{dI_-}{dz} = -\left(Q_1\sqrt{rg}\right)I_- + \left(-Q_2\sqrt{rg} + N\right)I_+,$$
$$-\frac{dI_+}{dz} = \left(Q_2\sqrt{rg} + N\right)I_- + \left(Q_1\sqrt{rg}\right)I_+. \tag{4}$$

where:

$$Q_{1/2}(s) = \tfrac{1}{2}\left(R_oY(s) \pm R_o{}^{-1}Z(s)\right), R_o = R_o{}^{-1} = \sqrt{\tfrac{L}{C}},$$
$$N(z) = \frac{1}{2}\frac{d}{dz}\ln\left(f_c(z)\right).$$

We simplify equations (4) by removing diagonal terms in (4) in the following way:

$$\frac{d}{dz}\left(I_-e^{-Q_1\alpha(z,z_1)}\right) = \left(Q_2\sqrt{rg} - N\right)e^{-Q_1\alpha(z,z_1)}I_+,$$
$$\frac{d}{dz}\left(I_+e^{-Q_1\alpha(z,z_1)}\right) = -\left(Q_2\sqrt{rg} + N\right)e^{-Q_1\alpha(z,z_1)}I_-, \tag{5}$$

where:

$$\alpha(y,x) = \int_x^y \sqrt{r(x)g(x)}\,dx.$$

B. Integral equations for dispersive nonuniform transmission line

Integrating the first of equations (5) from z to z_2 and the second one from z_1 to z, after simple but tedious manipulations we obtain the integral equations of the nonuniform dispersive transmission line:

$$I_-(z,s) = I_-(z_2,s)e^{-Q_1\alpha(z,z_1)} - \int_z^{z_1} Q_-(\xi,s)e^{-Q_1\alpha(\xi,z)}\,I_+(\xi,s)d\xi, \tag{6a}$$

$$I_+(z,s) = I_+(z_1,s)e^{-Q_1\alpha(z,z_1)} - \int_{z_1}^{z} Q_+(\xi,s)e^{-Q_1\alpha(z,\xi)}\,I_-(\xi,s)d\xi, \tag{6b}$$

where

$$Q_{+/-}(z,s) = \left(Q_2(s)\sqrt{r(z)g(z)} \pm N(z)\right).$$

The solutions of equations (6) have the following operator series form [4]:

$$I_-(z,s) = I_-(z_2,s)\sum_{i=0}^{\infty}\left(\mathfrak{I}^-(z_2,z)\mathfrak{I}^+(z,z_1)\right)^i e^{-Q_1\alpha(z_2,z)} - I_+(z_1,s)\sum_{i=0}^{\infty}\left(\mathfrak{I}^-(z_2,z)\mathfrak{I}^+(z,z_1)\right)^i\mathfrak{I}^-(z_2,z)\,e^{-Q_1\alpha(z,z_1)} \tag{7a}$$

$$I_+(z,s) = I_+(z_1,s)\sum_{i=0}^{\infty}\left(\mathfrak{I}^+(z,z_1)\mathfrak{I}^-(z_2,z)\right)^i e^{-Q_1\alpha(z,z_1)} - I_-(z_2,s)\sum_{i=0}^{\infty}\left(\mathfrak{I}^+(z,z_1)\mathfrak{I}^-(z_2,z)\right)^i\mathfrak{I}^+(z,z_1)\,e^{-Q_1\alpha(z_2,z)} \tag{7b}$$

where:

$$\mathfrak{I}^-(z_2,z)\{*\} = \int_z^{z_2} Q_-(\xi)\,e^{-Q_1\alpha(\xi,z)}\{*\}d\xi,$$
$$\mathfrak{I}^+(z,z_1)\{*\} = \int_{z_1}^{z} Q_+(\xi)e^{-Q_1\alpha(z,\xi)}\,\{*\}d\xi.$$

III. Scattering parameters for dispersive nonuniform transmission line

A. Scattering parameters for dispersive nonuniform transmission line

Now we substitute in equation (6a) $z = z_1$ and in (6b) $z = z_2$. As a result, we arrive at two-port equations of the transmission line expressed by current waves and scattering parameters in the following form:

$$b_1 = S_{11}a_1 + S_{12}a_2,$$
$$b_2 = S_{21}a_1 + S_{22}a_2, \tag{8}$$

where:

$$a_1 = I_+(z_1,s),\ a_2(s) = I_-(z_2,s),$$
$$b_1 = I_-(z_1,s),\ b_2(s) = I_+(z_2,s),$$
$$S_{11}(s) = \left[-\sum_{i=0}^{\infty}\left(\mathfrak{I}^-(z_2,z)\mathfrak{I}^+(z,z_1)\right)^i\mathfrak{I}^-(z_2,z)\,e^{-Q_1\alpha(z,z_1)}\right]_{z=z_1},$$
$$S_{22}(s) = \left[-\sum_{i=0}^{\infty}\left(\mathfrak{I}^+(z,z_1)\mathfrak{I}^-(z_2,z)\right)^i\mathfrak{I}^+(z,z_1)\,e^{-Q_1\alpha(z_2,z)}\right]_{z=z_2},$$
$$S_{12}(s) = S_{21}(s) = \left[\sum_{i=0}^{\infty}\left(\mathfrak{I}^-(z_2,z)\mathfrak{I}^+(z,z_1)\right)^i e^{-Q_1\alpha(z_2,z)}\right]_{z=z_1}. \tag{9}$$

The scattering parameters have the form of infinite series (9). Each term in these three series (9) is the integral of its predecessor. The integration of successive terms in these series can be done analytically or for more complex nonuniformities numerically.

B. Convergence of the series equivalent to scattering parameters for a dispersive nonuniform transmission line

Let us consider the first order approximation of the series (9). It means that we take terms in (9) for i = 0 only. Then we obtain relationships:

$$S_{11}^0(s) = \left[-\mathfrak{I}^-(z_2,z)e^{-Q_1\alpha(z,z_1)}\right]_{z=z_1} = -\int_{z_1}^{z_2} Q_-(\xi,s)e^{-2Q_1\alpha(\xi,z_1)}\,d\xi, \tag{10a}$$

$$S_{22}^{0}(s) = \left[-\Im^{+}(z_1, z)e^{-Q_1\alpha(z_2,z)}\right]_{z=z_2} = -\int_{z_1}^{z_2} Q_{+}(\xi, s)e^{-2Q_1\alpha(z_2,\xi)}\, d\xi, \qquad (10b)$$

$$S_{12}(s) = S_{21}(s) = \left[e^{-Q_1\alpha(z_2,z)}\right]_{z=z_1}. \qquad (10c)$$

Equations (10) are valuable results if the series (9) are rapidly converging. It can be shown that series (9) are no less rapidly convergent than the following two series:

$$S_{11}(s), S_{22}(s) \sim \left|\frac{Q_0^-}{2Q_1}\right| |B_1| + \left|\frac{(Q_0^-)^2 Q_0^+}{(2Q_1)^3}\right| |B_2| + \cdots, \qquad (11a)$$

$$S_{12}(s) \sim 1 + \left|\frac{Q_0^- Q_0^+}{(2Q_1)^2}\right| |C_2| + \left|\frac{(Q_0^- Q_0^+)^2}{(2Q_1)^4}\right| |C_4| \cdots, \qquad (11b)$$

where

$$Q_0^{\pm} = \max_{z,s} \left| Q_2(s) \pm \frac{N(z)}{\sqrt{r(z)g(z)}} \right|.$$

From estimates (11), it appears that the convergence of the series (9) is determined by the ratios $r_1 = |Q_0^- Q_0^+ / 4Q_1^2|$ and $r_2 = |Q_0^{\mp}/2Q_1|$, which are usually less than an one. This shows that the first order approximation of scattering parameters (10) may, in many cases be sufficient. We will show this in the example of the Bessel line.

IV. Scattering parameters for the Bessel transmission line

The PUL parameters of the Bessel transmission line are $Z(s)z^{\alpha}$ and $Y(s)z^{\beta}$. By substituting the above PUL to equations (10) and performing integrations we obtain:

$$So_{11}(s) = \frac{Q_2(s)}{2Q_1(s)}\left[e^{-2Q_1(s)q(z_2^{1/q} - z_1^{1/q})} - 1\right] + \frac{\alpha-\beta}{4} q e^{Q_1(s)qz_1^{1/q}}\left\{Ei\left(1, 2Q_1(s)qz_1^{1/q}\right) - Ei\left(1, 2Q_1(s)qz_{21}^{1/q}\right)\right\}, \qquad (12a)$$

$$So_{22}(s) = \frac{Q_2(s)}{2Q_1(s)}\left[e^{-2Q_1(s)q(z_2^{1/q} - z_1^{1/q})} - 1\right] - \frac{\alpha-\beta}{4} q e^{-Q_1(s)qz_2^{1/q}}\left\{Ei\left(1, -2Q_1(s)qz_1^{1/q}\right) - Ei\left(1, -2Q_1(s)qz_2^{1/q}\right)\right\}, \qquad (12b)$$

$$So_{12}(s) = So_{21}(s) = e^{-Q_1(s)q(z_2^{1/q} - z_1^{1/q})}, \qquad (12c)$$

where Ei(1,x) is an exponential integral and $q=2/(\alpha+\beta+2)$. Scattering parameters in the case of the Bessel line can be determined analytically. For comparative purposes, scattering parameters were calculated for the Bessel line for $\alpha = -1$ and $\beta = 1$. The exact parameter S_{11} for this case is :

$$S_{11}(s) = \frac{(FK_m(b) \cdot FI_p(a) - FI_p(b) \cdot FK_m(a)}{FI_m(a) \cdot FK_m(b) - FK_m(a) \cdot FI_p(b)},$$

$$FI_{p/m}(z) = (R_o^{-1} I_0(\gamma z) \pm Y_{co} I_1(\gamma z))\sqrt{z} R_o/2, \qquad (13)$$
$$FK_{p/m}(z) = (R_o^{-1} K_0(\gamma z) \pm Y_{co} K_1(\gamma z))\sqrt{z} R_o/2,$$
$$\gamma = \sqrt{Z(s)Y(s)},\ Y_{co} = \sqrt{\frac{Z(s)}{Y(s)}}, z_1 = a,\ z_2 = b,$$

where $I_n(z)$ and $K_n(z)$ are modified Bessel functions of the first and second kind respectively. The approximate scattering parameters for the Bessel line easily obtained from equations (12), where we need to substitute $\alpha = -1$, $\beta = 1$ and q=1.

V. Results

As an example we have considered a nonuniform (Bessel) interconnect with frequency dependent parameters shown in Fig.1.

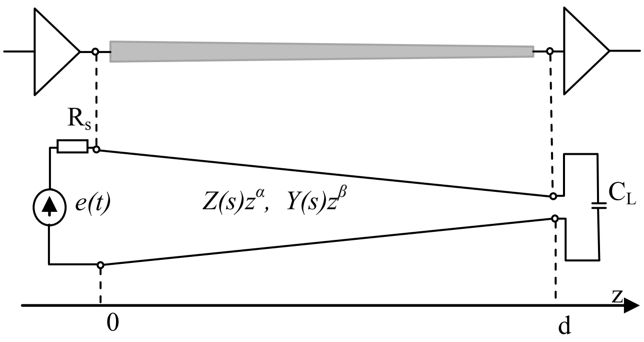

Fig.1 The considered system inverter-interconnect-inverter and its circuit model.

The longitudinal parameters of the interconnect Z(ω), Y(ω) depend on the frequency as follows:

$$Z(s) = R + sL + \left(0.1 + 10^{-4.5}\sqrt{s}\right) = Z_0(s) + Z_1(s),$$

$$Y(s) = G + sC\left(1 + \frac{\varepsilon_s/\varepsilon_\infty - 1}{1 + p\tau}\right) = Y_0(s) + Y_1(s),$$

where R=10Ω, L= 2nH, G=10μS, C=1pF, $\varepsilon_s = 4$, $\varepsilon_\infty = 1$, τ = 2ns, d=10cm.

The frequency dependence of the PUL parameters of interconnect R(ω), L(ω), G(ω), C(ω) are shown in Fig.2 and Fig.3. The above relationships of longitudinal parameters Z(ω),Y(ω) of the interconnect as functions of frequency were approximated by means of rational functions using very efficient algorithm-vector fitting [5]. Frequency characteristics of the modules $|S_{11}(\omega)|$, $|So_{11}(\omega)|$ for comparative purposes are shown in Fig.4. Moreover we simulated a circuit consisting of voltage pulse sources (of the trapezoid shape A=2V, $T_r = T_f$ =500ps, T_{on}=2ns) with source resistance R_s=150Ω and transmission line loaded by capacitor C_L=1pF (Fig.3).

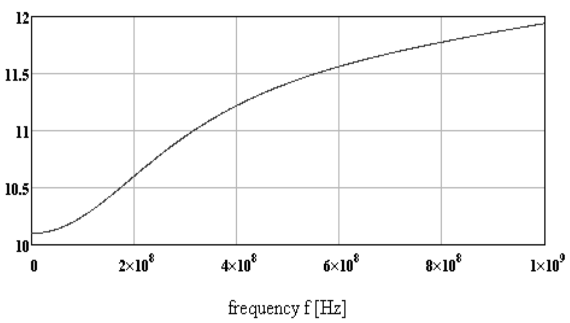

Fig.2 A typical frequency dependence of the real part of longitudinal impedance (resistance) of the line.

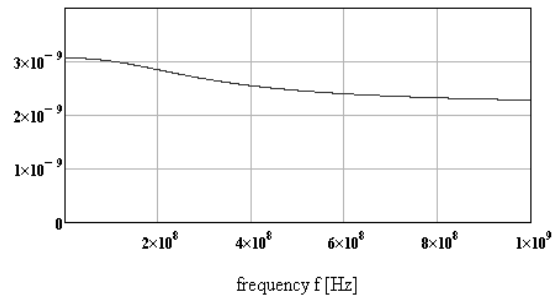

Fig.3 Transmission line inductance dependence on frequency.

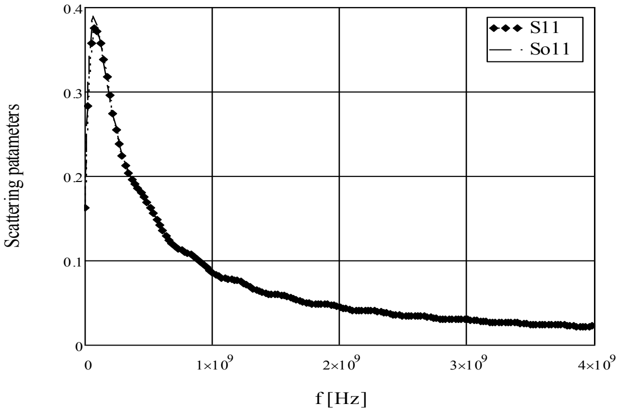

Fig.4 Dependence of scattering parameters: So_{11}- approximate and S_{11}-exact of nonuniform (Bessel) transmission line on frequency.

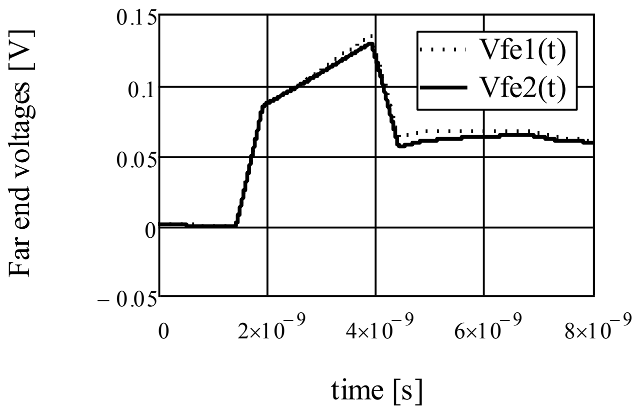

Fig.5 Near end voltages Vne1(t)-approximate and Vne2(t)-exact of the nonuniform (Bessel) transmission line.

The voltages at both ends of the Bessel transmission line were obtained based on the approximate (12) and exact (13) scattering parameters in the frequency domain and they were subsequently transformed (IFFT) to the time domain. The near end voltages of the considered system (Fig.1) are shown in Fig.5. Visible differences between the exact and approximate results occur in a steady state. The same effect can be observed in the case of voltages at the end of the line (Fig.6). It can be explained by the fact, that during the simulation a finite number of terms (first term in our case) contribute to S_{11}, S_{12}, S_{21}, S_{22}. For data in example considered by us, the ratios: $r_1 <$ 0.6 and , $r_2 < 0.5$ (for frequencies greater than 200MHz) , while

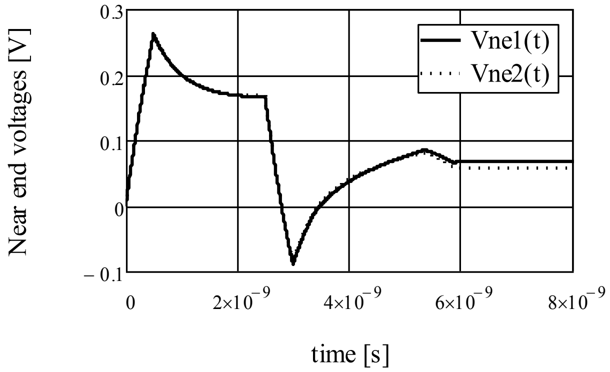

Fig.6 Far end voltages Vne1(t)-approximate and Vne2(t)-exact of the nonuniform (Bessel) transmission line.

calculated in example considered in [2] are $r_1 < 0.08$ and $r_2 <$ 0.012 It means that presented approach can be applied to the wider class of nonuniformities than in [2].

VI. CONCLUSIONS

We have shown that it is possible to generalize the approach based on the method of successive approximation for the case of a nonuniform transmission line with frequency dependent parameters. As a result, we obtain a closed form (meaning a first order approximation) of scattering parameters of nonuniform transmission line in frequency domain. In the case of a transmission line with $r_1 << 1$ and $r_2 << 1$, an approximation is satisfactory. Equations (10) allow us to determine the approximate scattering parameters for nonuniform lines by integrating analytically (such as in the case of the Bessel lines) or numerically and applying the approximation by rational functions using a vector fitting algorithm. Compared with the approach based on dyadic Green's function and parametric macromodeling applied to weakly nonuniform transmission lines [2,3] the presented approach is simpler. The presented approach permits the implementation of the model in the SPICE program.

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